The forces and displacements in Eq. (7) can be presented in a more tractable form. For instance, the column matrix of the force vectors can be put in the sequence as $\{F_{x1},F_{y1},F_{z1}, M_{x1},M_{y1},M_{z1},F_{x2},\ldots,M_{z2}\}^T$. The square array of elements in the stiffness matrix must then be rearranged accordingly, but the explicit form is too lengthy to be included. It can be done, however, much more easily when all terms in that stiffness matrix have been numerically evaluated.

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Generalized Gram-Charlier Method for Curve-Fitting Statistical Data

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Nomenclature

A_{pq}	=	matrix (3 × 3) of constants [see Eq. (3)]
B_{ij}	=	matrix (3 × 3) of constants [see Eq. (6)]
$C_{ij}(x_2)$	=	matrix (3 × 3) of numbers dependent on x_2 [see Eq. (2)]
C_{ijkl}	=	matrix (3 × 3 × 3 × 3) of constants [see Eq. (6)]
D_{pq}	=	matrix (3 × 3) of constants [see Eq. (5)]
E	=	square error [see Eq. (7)]
F_α	=	trial functions [see Eq. (6)]
$M_{ij}(x_2)$	=	matrix (3 × 3) of numbers [see Eqs. (4) and (5)]
r_i	=	three-vector position difference
R_{ij}	=	covariance tensor [see Eq. (1)]
$R_{\alpha\alpha}$	=	diagonal elements of R_{ij} (no sum)
x_i,x'_i	=	position vectors of measuring probes
$W(r)$	=	weight function [see Eq. (7)]
$\varphi(\mathbf{r},x_2)$	=	Gaussian distribution [see Eq. (3)]
$\psi(\mathbf{r},x_2)$	=	Gaussian distribution [see Eq. (5)]

THE method described below arose from the requirement to fit analytic forms to velocity covariances measured in the turbulent wake of a circular cylinder.[1] The covariance is defined by

$$R_{ij}(\mathbf{r},x_2) = u_i(\mathbf{x})u_j(\mathbf{x}') \quad (i,j = 1,2,3) \tag{1}$$

where $\mathbf{r} = \mathbf{x}' - \mathbf{x}$ is the three-vector separation of measuring probes located at $\mathbf{x}$ and $\mathbf{x}'$. Grant measured R_{ij} defined as in Eq. (1) for $i = j = 1, 2, 3$ for various $\mathbf{r}$, x_2 values. His measurements generated 49 curves.

Early attempts by the author to fit analytic expressions to the data were based on a series of multidimensional Hermite polynomials, e.g.,

$$R_{ij}(\mathbf{r},x_2) = C_{ij}(x_2) \sum_{m,n=0}^{k} \frac{\partial^{m+n}}{\partial r_m\,\partial r_n}\,\varphi(\mathbf{r},x_2) \tag{2}$$

where

$$\varphi(\mathbf{r},x_2) = \exp[-A_{pq}(x_2)r_pr_q], \quad A_{pq} \geqslant 0 \tag{3}$$

This scheme attempts to construct a model from the Gaussian distribution φ and its first k moments.

This approach proved unsatisfactory for the turbulence data. The difficulty was the determination of analytic approximations uniformly valid over a wide range of r_i separations; to be specific, good matches for small r_i decayed too rapidly as r_i increased to the order of half the wake thickness. Increasing the value of k from 4 to 8 showed, as indicated possible by Cramer,[2] little improvement.

Reconsideration of the physics of turbulent flows suggested a "two-component" model. The reasoning, similar to that of Grant[1] and Townsend,[3] is that the small-scale motions are nearly Gaussian whereas the large-scale motions surely are not. Hence, the model chosen used φ and its derivatives to represent the large eddies and a pure exponential to describe the small eddies, e.g.,

$$R_{ij}(\mathbf{r},x_2) = C_{ij}(x_2) \sum_{m,n=0}^{k} \frac{\partial^{m+n}}{\partial r_m\,\partial r_n}\,\varphi(\mathbf{r},x_2) + M_{ij}(x_2)\psi(\mathbf{r},x_2) \tag{4}$$

where φ is defined by (3), and

$$\psi(\mathbf{r},x_2) = \exp[-D_{pq}r_pr_q], \quad D_{pq} \geqslant 0, \quad M_{ij} = \mathbf{1} - C_{ij} \tag{5}$$

Applying (4) and (5) yielded, for the 49 curves measured by Grant, errors of less than 10% for the first five moments and pointwise errors generally of less than 10%. This is more than adequate due to inherent difficulties in turbulence measurements wherein 10–20% errors are customary.

The largest errors in quantities usually considered significant in turbulent flows occurred in the Taylor microscale, which is closely associated with the curvature at the r_i origin. This was not a serious fault for the immediate problem of determining large eddy structure, which is detailed elsewhere.[4,5]

For applications in which the Taylor microscale is important, it appears that generalization of ψ to higher-order Hermitian polynomials is the simplest approach. This is consistent with the idea that ψ describes the small-scale eddies that determine the Taylor microscale.

A note on computation techniques as used on the IBM 7074 may be useful. The error criterion used to determine the best match, e.g., best values of D_{pq}, C_{ij}, A_{pq}, was to minimize the integrated square error. Grant's data included only the trace of the correlation tensor, hence $i = j$. For computational simplicity the programed forms of (4) and (5) were taken as

$$F_\alpha = C_{\alpha\alpha}(1 + B_{ij}r_ir_j + C_{ijkl}r_ir_jr_kr_l)\exp[-A_{pq}r_pr_q] + (1 - C_{\alpha\alpha})\exp[-D_{pq}r_pr_q] \tag{6}$$

where F_α ($\alpha = 1,2,3$) is the trial match for $R_{\alpha\alpha}$ (no sum on repeated Greek indices).

Since Grant's data consisted entirely of $R_{\alpha\alpha}$ measurements and predominantly of probe separations along the r_i axes, the form (6) was programed in a considerably simplified version. The A_{pq} and D_{pq} were taken to be diagonal tensors. The odd-ordered derivatives of φ were dropped as well as cross derivatives. The basis for these simplifications is the lack of data at separations in more than one r_i direction. Therefore, B_{ij} and C_{ij} also were taken to be diagonal and dependent only on x_2.

Since Grant found three basic types of correlations, i.e., positive, single zero, and double zero, slightly different programs were used in each case in the initial investigation.[5] Zeroes and/or zero slope points of $R_{\alpha\alpha}$ were used to reduce computer time. Initial values were picked for $C_{\alpha\alpha}$, A_{ij}, and D_{ij} and automatically varied to minimize the square error E defined by

$$E = \int_{-\infty}^{\infty} (R_{\alpha\alpha} - F_\alpha)^2 W(r)\,dr \tag{7}$$

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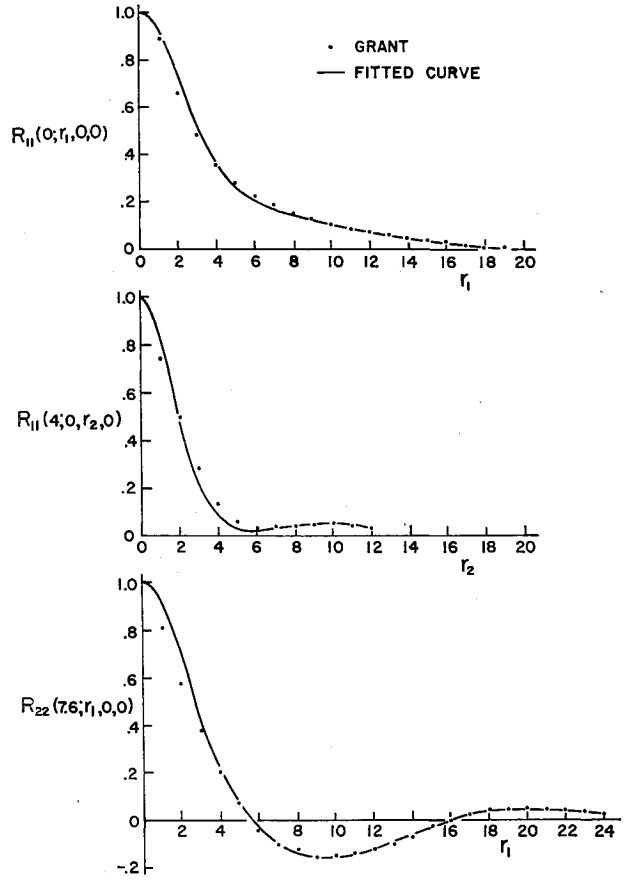

Fig. 1 Sample curve-fits to $R_{\alpha\alpha}$.

where the integral was evaluated by the trapezoidal and Simpson's rules. Here $W(r)$ is some weight function which in Ref. 5 was simply $(R_{\alpha\alpha})^{-2}$.

In some cases matrices were generated which were very poorly conditioned and required special treatment; also in some cases the programs were caught in a local minimum of E. The later problem was solved by computer plots of $R_{\alpha\alpha}$ and F_α for minimum E given different initial conditions for the undetermined parameters in Eq. (6). In addition, the first five moments of $R_{\alpha\alpha}$ and F_α were computed for comparison. This allowed better initial conditions for repeat runs. Sample curve fits are given in Fig. 1. More comprehensive results are given in Ref. 5.

Recently, the author has had success with $W(r) = \exp[-A_{\beta\beta}r_\beta r_\beta]$. This latter procedure begins by estimating $D_{\beta\beta}$ and forming the new "covariance" $\tilde{R}_{\alpha\alpha} = R_{\alpha\alpha} - (1 - C_{\alpha\alpha})\psi(\mathbf{r},x_2)$ and minimizing (7) for $\tilde{R}_{\alpha\alpha}$. This approach allows the use of a single algorithm for all types of curves, and computer run time is greatly reduced. The F_α curve-fits are essentially the same with the new techniques as the old. This procedure is a modification of a method given by Hildebrand.[6]

It should be remarked that the forms actually computed, i.e., (6), did not in every case satisfy the non-negative-definite property of covariances.[7] The reasons for this are simply experimental errors in the data.

Resolution of this difficulty, though not crucial in this investigation[5] because the F_α were covariances to within empirical error, will require two improvements. First, more accurate measurements should insure that the data satisfy the non-negative criterion. Second, the curve-fitting method should be modified so that it automatically selects only that portion of the data which will satisfy the non-negative-definite property.

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